The Tetrahedron algebra, the Onsager algebra, and the $\mathfrak{sl}_2$ loop algebra

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Abstract

Let $\mathbb{K}$ denote a field with characteristic 0 and let T denote an indeterminate. We give a presentation for the three-point loop algebra $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$ via generators and relations. This presentation displays S_4-symmetry. Using this presentation we obtain a decomposition of the above loop algebra into a direct sum of three subalgebras, each of which is isomorphic to the Onsager algebra.

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1 Introduction

In a seminal paper by Onsager [31] the free energy of the two dimensional Ising model was computed exactly. In that paper a certain infinite dimensional Lie algebra was introduced; this is now called the Onsager algebra and we will denote it by O. Over the years O has been investigated many times in connection with solvable lattice models [1], [2], [3], [5], [6], [7], [8], [15], [16], [22], [26], [27], [28], [30], [39] representation theory [13], [14], [21] Kac-Moody algebras [12], [32], [33] tridiagonal pairs [23], [24], [36], [37], [38] and partially orthogonal polynomials [19], [20]. Let us recall some results on the mathematical side. In [13], [14] Davies classified the irreducible finite dimensional O-modules. In [32] Perk showed that O has a presentation involving generators A, B and relations

$$\begin{aligned} [A,[A,[A,B]]] &= 4[A,B], \\ [B,[B,[B,A]]] &= 4[B,A]. \end{aligned}$$

In [33] Roan obtained an injection from O into the loop algebra $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$ where $\mathbb{K}$ denotes a field with characteristic 0 and T denotes an indeterminate. In [12] Date and Roan used this injection to link the representation theories of O and $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$.

In this paper we investigate further the relationship between O and $\mathfrak{sl}_2$ loop algebras. But instead of working with $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$ we will work with $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$. This

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algebra appears in [11] and [17, Section 4.3]; see also [9], [10], [34], [35]. Our first main result is a presentation for $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$ via generators and relations. To obtain this presentation we define a Lie algebra $\boxtimes$ using generators and relations, and eventually show that $\boxtimes$ is isomorphic to $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$. We remark that our presentation of $\boxtimes$ displays an S_4-symmetry. In our second main result we use the above presentation to get a decomposition of $\boxtimes$ into a direct sum of three subalgebras, each of which is isomorphic to O.

We now give a formal definition of $\boxtimes$, followed by a more detailed description of our results.

Definition 1.1 Let $\boxtimes$ denote the Lie algebra over $\mathbb{K}$ that has generators

$$\{X_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad\qquad \mathbb{I} = \{0, 1, 2, 3\} \tag{1}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$X_{ij} + X_{ji} = 0.$$

(ii) For mutually distinct $h, i, j \in \mathbb{I}$,

$$[X_{hi}, X_{ij}] = 2X_{hi} + 2X_{ij}.$$

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,

$$[X_{hi}, [X_{hi}, [X_{hi}, X_{jk}]]] = 4[X_{hi}, X_{jk}].$$

We call $\boxtimes$ the *Tetrahedron algebra*.

In this paper we will prove:

- For mutually distinct $h, i, j \in \mathbb{I}$ the elements X_{hi}, X_{ij}, X_{jh} form a basis for a subalgebra of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$.
- For mutually distinct $h, i, j, k \in \mathbb{I}$ the subalgebra of $\boxtimes$ generated by X_{hi}, X_{jk} is isomorphic to O.
- For distinct $r, s \in \mathbb{I}$ the subalgebra of $\boxtimes$ generated by

  $$\{X_{ij} \mid i, j \in \mathbb{I}, i \neq j, (i, j) \neq (r, s), (i, j) \neq (s, r)\}$$

  is isomorphic to $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$.
- $\boxtimes$ is isomorphic to $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$.
- Let Ω (resp. Ω') (resp. Ω'') denote the subalgebra of $\boxtimes$ generated by X_{12}, X_{03} (resp. X_{23}, X_{01}) (resp. X_{31}, X_{02}). By the second bullet above, each of $\Omega, \Omega', \Omega''$ is isomorphic to O. Then the $\mathbb{K}$-vector space $\boxtimes$ satisfies

  $$\boxtimes = \Omega + \Omega' + \Omega'' \qquad\qquad \text{(direct sum)}.$$

2 An S_4-action on $\boxtimes$

In this section we describe how the symmetric group S_4 acts on the Tetrahedron algebra as a group of automorphisms. We will also review some notational conventions.

We identify S_4 with the group of permutations of $\mathbb{I}$. We denote elements of S_4 using the cycle notation. For example (123) denotes the element of S_4 that sends $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $0 \mapsto 0$. The group S_4 acts on the set of generators for $\boxtimes$ by permuting the indices. Thus each $\tau \in S_4$ sends

$$X_{ij} \quad \mapsto \quad X_{i^\tau j^\tau} \qquad\qquad (i, j \in \mathbb{I},\ i \neq j). \tag{2}$$

This action leaves invariant the defining relations for $\boxtimes$ and therefore induces a group homomorphism $S_4 \to \mathrm{Aut}(\boxtimes)$, where $\mathrm{Aut}(\boxtimes)$ denotes the group of automorphisms of $\boxtimes$. This gives an action of S_4 on $\boxtimes$ as a group of automorphisms. For notational convenience we give certain elements of S_4 special names:

$$\prime = (123), \qquad\qquad \omega = (13), \qquad\qquad d = (13)(02), \tag{3}$$

$$\downarrow = (12), \qquad\qquad \Downarrow = (03), \qquad\qquad * = (01)(23). \tag{4}$$

The same notation will be used for the images of these elements in $\mathrm{Aut}(\boxtimes)$. For example

$$X'_{01} = X_{02}, \qquad\qquad X'_{02} = X_{03}, \qquad\qquad X'_{03} = X_{01}, \tag{5}$$

$$X'_{12} = X_{23}, \qquad\qquad X'_{23} = X_{31}, \qquad\qquad X'_{31} = X_{12}. \tag{6}$$

Throughout this paper all group actions are assumed to be from the right; this means that when we apply a product $\tau\sigma$ we apply τ first and then σ. For example

$$X'^{*}_{12} = (X'_{12})^* = X^*_{23} = X_{32}.$$

The following subgroups of S_4 will play a role in our discussion. Observe that $\prime, \omega$ generate a subgroup of S_4 that is isomorphic to the symmetric group S_3. Observe that $\downarrow, \Downarrow, *$ generate a subgroup of S_4 that is isomorphic to the dihedral group D_4. Observe that ω, d generate a subgroup of S_4 that is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Note 2.1 In what follows we will discuss several Lie algebras and their relationship to $\boxtimes$. These other Lie algebras possess some automorphisms that we will denote by $\prime, \omega, d, \downarrow, \Downarrow, *$. We trust that for any given automorphism, the algebra on which it acts will be clear from the context.

3 The Lie algebra $\mathfrak{sl}_2$

In this section we discuss the Lie algebra $\mathfrak{sl}_2$ and its relationship to $\boxtimes$.

Definition 3.1 We let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{K}$ that has a basis e, f, h and Lie bracket

$$[h, e] = 2e, \qquad\qquad [h, f] = -2f, \qquad\qquad [e, f] = h.$$

Lemma 3.2 *$\mathfrak{sl}_2$ is isomorphic to the Lie algebra over $\mathbb{K}$ that has basis X, Y, Z and Lie bracket*

$$[X,Y] = 2X + 2Y, \qquad [Y,Z] = 2Y + 2Z, \qquad [Z,X] = 2Z + 2X. \tag{7}$$

An isomorphism with the presentation in Definition 3.1 is given by

$$X \rightarrow 2e - h, \qquad Y \rightarrow -2f - h, \qquad Z \rightarrow h.$$

The inverse of this isomorphism is given by

$$e \rightarrow (X+Z)/2, \qquad f \rightarrow -(Y+Z)/2, \qquad h \rightarrow Z.$$

Proof: One readily checks that each map is a homomorphism of Lie algebras and that the maps are inverses. It follows that each map is an isomorphism of Lie algebras. $\square$

Note 3.3 For notational convenience, for the rest of this paper we identify the copy of $\mathfrak{sl}_2$ given in Definition 3.1 with the copy given in Lemma 3.2, via the isomorphism given in Lemma 3.2.

We now describe two automorphisms of $\mathfrak{sl}_2$ that will play a role later in the paper.

Lemma 3.4 *The following (i), (ii) hold.*

(i) There exists an automorphism $'$ of $\mathfrak{sl}_2$ such that

$$X' = Y, \qquad Y' = Z, \qquad Z' = X. \tag{8}$$

(ii) There exists an automorphism ω of $\mathfrak{sl}_2$ such that

$$X^\omega = -Y, \qquad Y^\omega = -X, \qquad Z^\omega = -Z.$$

Proof: (i) Define the linear transformation $' : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$ so that (8) holds. We observe that $'$ is a bijection. Using (7) we find $[u,v]' = [u', v']$ for all $u, v \in \mathfrak{sl}_2$.
(ii) Similar to the proof of (i) above. $\square$

Note 3.5 Referring to Lemma 3.4, the automorphism $'$ has order 3 and each of $\omega, {}'\omega$ has order 2. Therefore $', \omega$ generate a subgroup of $\mathrm{Aut}(\mathfrak{sl}_2)$ that is isomorphic to S_3.

Proposition 3.6 *Let h, i, j denote mutually distinct elements of $\mathbb{I}$. Then there exists a unique Lie algebra homomorphism from $\mathfrak{sl}_2$ to $\boxtimes$ that sends*

$$X \rightarrow X_{hi}, \qquad Y \rightarrow X_{ij}, \qquad Z \rightarrow X_{jh}.$$

Proof: By Definition 1.1(ii) the elements X_{hi}, X_{ij}, X_{jh} satisfy the defining relations (7) for $\mathfrak{sl}_2$. Therefore the homomorphism exists. The homomorphism is unique since X, Y, Z form a basis for $\mathfrak{sl}_2$. $\square$

Note 3.7 In Section 12 we will show that the homomorphism in Proposition 3.6 is an injection.

Definition 3.8 By the *standard homomorphism* from $\mathfrak{sl}_2$ to $\boxtimes$ we mean the homomorphism in Proposition 3.6 with $(h, i, j) = (1, 2, 3)$.

We finish this section with a comment.

Lemma 3.9 *The following diagrams commute:*

$$\begin{tikzcd} \mathfrak{sl}_2 \arrow[r, "\text{st. hom}"] \arrow[d, "\prime"'] & \boxtimes \arrow[d, "\prime"] \\ \mathfrak{sl}_2 \arrow[r, "\text{st. hom}"] & \boxtimes \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{sl}_2 \arrow[r, "\text{st. hom}"] \arrow[d, "\omega"'] & \boxtimes \arrow[d, "\omega"] \\ \mathfrak{sl}_2 \arrow[r, "\text{st. hom}"] & \boxtimes \end{tikzcd}$$

4 The Onsager algebra

In this section we discuss the Onsager algebra and its relationship to the Tetrahedron algebra. Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$.

Definition 4.1 *[31]* Let O denote the Lie algebra over $\mathbb{K}$ with basis A_m, G_l, $m \in \mathbb{Z}$, $l \in \mathbb{N}$ and Lie bracket

$$\begin{aligned} [A_l, A_m] &= 2G_{l-m} \qquad\qquad l > m, \\ [G_l, A_m] &= A_{m+l} - A_{m-l}, \\ [G_l, G_m] &= 0. \end{aligned}$$

We call O the *Onsager algebra*.

Note 4.2 In Definition 4.1 our notation is a bit nonstandard. The elements called A_m, G_l in Definition 4.1 correspond to the elements called $A_m/2, G_l/2$ in [12]. We make this adjustment for notational convenience.

Lemma 4.3 *[32] O is isomorphic to the Lie algebra over $\mathbb{K}$ that has generators A, B and relations*

$$[A, [A, [A, B]]] = 4[A, B], \tag{9}$$

$$[B, [B, [B, A]]] = 4[B, A]. \tag{10}$$

An isomorphism with the presentation in Definition 4.1 is given by

$$A \rightarrow A_0, \qquad\qquad B \rightarrow A_1.$$

Note 4.4 In what follows we identify the copy of O given in Definition 4.1 with the copy given Lemma 4.3, via the isomorphism in Lemma 4.3.

We now describe three automorphisms of O that will play a role in our discussion.

Lemma 4.5 *The following (i)–(iii) hold.*

(i) There exists an automorphism $\downarrow$ of O such that $A^{\downarrow} = -A$ and $B^{\downarrow} = B$. We have

$$A_m^{\downarrow} = (-1)^{m-1} A_m, \qquad G_l^{\downarrow} = (-1)^l G_l \qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}.$$

(ii) There exists an automorphism $\Downarrow$ of O such that $A^{\Downarrow} = A$ and $B^{\Downarrow} = -B$. We have

$$A_m^{\Downarrow} = (-1)^m A_m, \qquad G_l^{\Downarrow} = (-1)^l G_l \qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}.$$

(iii) There exists an automorphism $*$ of O such that $A^* = B$ and $B^* = A$. We have

$$A_m^* = A_{1-m}, \qquad G_l^* = -G_l \qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}.$$

Proof: In each case the map is invertible and respects the defining relations for O. $\square$

Note 4.6 The automorphisms $\downarrow, \Downarrow, *$ from Lemma 4.5 generate a subgroup of $\mathrm{Aut}(O)$ that is isomorphic to D_4.

Proposition 4.7 *Let h, i, j, k denote mutually distinct elements of $\mathbb{I}$. Then there exists a unique Lie algebra homomorphism from O to $\boxtimes$ that sends*

$$A \to X_{hi}, \qquad B \to X_{jk}.$$

Proof: By Definition 1.1(iii) the elements X_{hi}, X_{jk} satisfy the defining relations (9), (10) for O. Therefore the homomorphism exists. The homomorphism is unique since A, B together generate O. $\square$

Note 4.8 In Section 12 we will show that the homomorphism in Proposition 4.7 is an injection.

Definition 4.9 By the *standard homomorphism* from O to $\boxtimes$ we mean the homomorphism from Proposition 4.7 with $(h, i, j, k) = (1, 2, 0, 3)$.

We finish this section with a comment.

Lemma 4.10 *The following diagrams commute:*

$$\begin{tikzcd} O \arrow[r, "\text{st. hom}"] \arrow[d, "\downarrow"'] & \boxtimes \arrow[d, "\downarrow"] \\ O \arrow[r, "\text{st. hom}"] & \boxtimes \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, "\text{st. hom}"] \arrow[d, "\Downarrow"'] & \boxtimes \arrow[d, "\Downarrow"] \\ O \arrow[r, "\text{st. hom}"] & \boxtimes \end{tikzcd} \qquad \begin{tikzcd} O \arrow[r, "\text{st. hom}"] \arrow[d, "*"'] & \boxtimes \arrow[d, "*"] \\ O \arrow[r, "\text{st. hom}"] & \boxtimes \end{tikzcd}$$

5 The $\mathfrak{sl}_2$ loop algebra

In this section we discuss the $\mathfrak{sl}_2$ loop algebra and its relationship to $\boxtimes$.

Definition 5.1 Let T denote an indeterminate. Let $\mathbb{K}[T, T^{-1}]$ denote the $\mathbb{K}$-algebra consisting of all Laurant polynomials in T that have coefficients in $\mathbb{K}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{K}$ consisting of the $\mathbb{K}$-vector space $\mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \qquad a, b \in \mathbb{K}[T, T^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ *loop algebra.*

The $\mathfrak{sl}_2$ loop algebra is related to the Kac-Moody algebra associated with the Cartan matrix

$$A := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

This is made clear in the following lemma.

Lemma 5.2 *[25, p. 100] The loop algebra $L(\mathfrak{sl}_2)$ is isomorphic to the Lie algebra over $\mathbb{K}$ that has generators e_i, f_i, h_i, $i \in \{0, 1\}$ and the following relations:*

$$\begin{aligned}
h_0 + h_1 &= 0, \\
[h_i, e_j] &= A_{ij} e_j, \\
[h_i, f_j] &= -A_{ij} f_j, \\
[e_i, f_j] &= \delta_{ij} h_j, \\
[e_i, [e_i, [e_i, e_j]]] &= 0, \qquad i \neq j, \\
[f_i, [f_i, [f_i, f_j]]] &= 0, \qquad i \neq j.
\end{aligned}$$

An isomorphism is given by

$$\begin{aligned}
e_1 &\to e \otimes 1, & f_1 &\to f \otimes 1, & h_1 &\to h \otimes 1, \\
e_0 &\to f \otimes T, & f_0 &\to e \otimes T^{-1}, & h_0 &\to -h \otimes 1.
\end{aligned}$$

We now give a second presentation for $L(\mathfrak{sl}_2)$.

Lemma 5.3 *$L(\mathfrak{sl}_2)$ is isomorphic to the Lie algebra over $\mathbb{K}$ that has generators X_i, Y_i, Z_i, $i \in \{0, 1\}$ and the following relations.*

$$\begin{aligned}
Z_0 + Z_1 &= 0, \\
[X_i, Y_i] &= 2X_i + 2Y_i, \\
[Y_i, Z_i] &= 2Y_i + 2Z_i, \\
[Z_i, X_i] &= 2Z_i + 2X_i, \\
[Y_i, X_j] &= 2Y_i + 2X_j \qquad i \neq j, \\
[X_i, [X_i, [X_i, X_j]]] &= 4[X_i, X_j], \qquad i \neq j, \\
[Y_i, [Y_i, [Y_i, Y_j]]] &= 4[Y_i, Y_j], \qquad i \neq j.
\end{aligned}$$

An isomorphism with the presentation in Lemma 5.2 is given by

$$X_i \;\to\; 2e_i - h_i, \qquad\qquad Y_i \;\to\; -2f_i - h_i, \qquad\qquad Z_i \;\to\; h_i.$$

The inverse of this isomorphism is given by

$$e_i \;\to\; (X_i + Z_i)/2, \qquad\qquad f_i \;\to\; -(Y_i + Z_i)/2, \qquad\qquad h_i \;\to\; Z_i.$$

Proof: One routinely checks that each map is a homomorphism of Lie algebras and that the maps are inverses. It follows that each map is an isomorphism of Lie algebras. □

Note 5.4 In what follows we identify the copies of $L(\mathfrak{sl}_2)$ given in Definition 5.1, Lemma 5.2 and Lemma 5.3, via the isomorphisms given in Lemma 5.2 and Lemma 5.3.

We now describe two automorphisms of $L(\mathfrak{sl}_2)$ that will play a role in our discussion.

Lemma 5.5 *The following (i), (ii) hold.*

(i) There exists an automorphism ω of $L(\mathfrak{sl}_2)$ such that

$$X_i^\omega = -Y_i, \qquad\qquad Y_i^\omega = -X_i, \qquad\qquad Z_i^\omega = -Z_i \qquad\qquad i \in \{0,1\}.$$

(ii) There exists an automorphism d of $L(\mathfrak{sl}_2)$ such that

$$X_i^d = X_j, \qquad\qquad Y_i^d = Y_j, \qquad\qquad Z_i^d = Z_j \qquad\qquad i,j \in \{0,1\}, \quad i \neq j.$$

Note 5.6 The automorphisms ω, d from Lemma 5.5 generate a subgroup of $\mathrm{Aut}(L(\mathfrak{sl}_2))$ that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 5.7 *Let h, i, j, k denote mutually distinct elements of $\mathbb{I}$. Then there exists a unique Lie algebra homomorphism from $L(\mathfrak{sl}_2)$ to $\boxtimes$ that sends*

$$\begin{aligned} X_1 &\to X_{hi}, & \qquad Y_1 &\to X_{ij}, & \qquad Z_1 &\to X_{jh}, \\ X_0 &\to X_{jk}, & \qquad Y_0 &\to X_{kh}, & \qquad Z_0 &\to X_{hj}. \end{aligned}$$

Proof: Comparing the relations given in Lemma 5.3 with the relations given in Definition 1.1, we find the homomorphism exists. This homomorphism is unique since X_i, Y_i, Z_i, $i \in \{0,1\}$ is a generating set for $L(\mathfrak{sl}_2)$. □

Note 5.8 In Section 12 we will show that the homomorphism in Proposition 5.7 is an injection.

Definition 5.9 By the *standard homomorphism* from $L(\mathfrak{sl}_2)$ to $\boxtimes$ we mean the homomorphism from Proposition 5.7 with $(h, i, j, k) = (1, 2, 3, 0)$.

We finish this section with a comment.

Lemma 5.10 *The following diagrams commute:*

$$\begin{array}{ccc} L(\mathfrak{sl}_2) & \xrightarrow{\text{st. hom}} & \boxtimes \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ L(\mathfrak{sl}_2) & \xrightarrow{\text{st. hom}} & \boxtimes \end{array} \qquad\qquad \begin{array}{ccc} L(\mathfrak{sl}_2) & \xrightarrow{\text{st. hom}} & \boxtimes \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ L(\mathfrak{sl}_2) & \xrightarrow{\text{st. hom}} & \boxtimes \end{array}$$

6 The three-point $\mathfrak{sl}_2$ loop algebra

In this section we consider an extension of $L(\mathfrak{sl}_2)$ that we call $L(\mathfrak{sl}_2)^+$. This algebra is defined as follows.

Definition 6.1 We abbreviate $\mathcal{A}$ for the $\mathbb{K}$-algebra $\mathbb{K}[T, T^{-1}, (T-1)^{-1}]$, where T is indeterminate. Let $L(\mathfrak{sl}_2)^+$ denote the Lie algebra over $\mathbb{K}$ consisting of the $\mathbb{K}$-vector space $\mathfrak{sl}_2 \otimes \mathcal{A}$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \qquad a, b \in \mathcal{A}. \tag{11}$$

Following [11] we call $L(\mathfrak{sl}_2)^+$ the *three-point* $\mathfrak{sl}_2$ *loop algebra.*

Our next goal is to display a basis for $L(\mathfrak{sl}_2)^+$. We start with an observation.

Lemma 6.2 *There exists a unique $\mathbb{K}$-algebra automorphism $'$ of $\mathcal{A}$ such that $T' = 1 - T^{-1}$. This automorphism has order 3 and satisfies*

$$T'' = (1-T)^{-1}, \qquad TT' = T - 1, \tag{12}$$

$$T'T'' = T' - 1, \qquad T''T = T'' - 1. \tag{13}$$

Lemma 6.3 *The following is a basis for the $\mathbb{K}$-vector space $\mathcal{A}$:*

$$\{1\} \cup \{T^i, (T')^i, (T'')^i \mid i \in \mathbb{N}\}. \tag{14}$$

Proof: We first claim that the elements (14) span $\mathcal{A}$. Let $\mathcal{A}_1$ denote the subspace of $\mathcal{A}$ spanned by the elements (14). Using the data in Lemma 6.2 we find $\mathcal{A}_1$ is closed under multiplication and contains the generators $T, T^{-1}, (T-1)^{-1}$ of $\mathcal{A}$. Therefore $\mathcal{A}_1 = \mathcal{A}$ and our claim is proved. It is routine to check that the elements (14) are linearly independent and hence form a basis for $\mathcal{A}$. $\square$

Lemma 6.4 *The following is a basis for the $\mathbb{K}$-vector space $L(\mathfrak{sl}_2)^+$:*

$$\begin{aligned}
&\{X \otimes 1, Y \otimes 1, Z \otimes 1\} \cup \{X \otimes T^i, Y \otimes T^i, Z \otimes T^i \mid i \in \mathbb{N}\} \\
&\cup \ \{X \otimes (T')^i, Y \otimes (T')^i, Z \otimes (T')^i \mid i \in \mathbb{N}\} \\
&\cup \ \{X \otimes (T'')^i, Y \otimes (T'')^i, Z \otimes (T'')^i \mid i \in \mathbb{N}\}.
\end{aligned}$$

Here X, Y, Z is the basis for $\mathfrak{sl}_2$ given in Lemma 3.2.

Proof: Combine Definition 6.1 and Lemma 6.3. $\square$

We now consider how $\boxtimes$ is related to $L(\mathfrak{sl}_2)^+$.

Proposition 6.5 *There exists a unique Lie algebra homomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$ such that*

$$\begin{aligned}
X_{12}^\sigma &= X \otimes 1, & X_{03}^\sigma &= Y \otimes T + Z \otimes (T-1), \\
X_{23}^\sigma &= Y \otimes 1, & X_{01}^\sigma &= Z \otimes T' + X \otimes (T'-1), \\
X_{31}^\sigma &= Z \otimes 1, & X_{02}^\sigma &= X \otimes T'' + Y \otimes (T''-1).
\end{aligned}$$

Here X, Y, Z is the basis for $\mathfrak{sl}_2$ given in Lemma 3.2.

Proof: Using (7) and (11)–(13) we find that in the above equations the expressions on the right-hand side satisfy the defining relations for $\boxtimes$ given in Definition 1.1. Therefore the homomorphism exists. The homomorphism is unique since $X_{12}, X_{23}, X_{31}, X_{01}, X_{02}, X_{03}$ is a generating set for $\boxtimes$. $\square$

Note 6.6 In Section 11 we will show that the homomorphism in Proposition 6.5 is an isomorphism.

We now introduce several maps that will be useful later in the paper.

Lemma 6.7 There exists an automorphism $\prime$ of $L(\mathfrak{sl}_2)^+$ that satisfies

$$(u \otimes a)' = u' \otimes a' \qquad\qquad u \in \mathfrak{sl}_2, \qquad a \in \mathcal{A}, \tag{15}$$

where u' is from Lemma 3.4(i) and a' is from Lemma 6.2. This automorphism has order 3.

Proof: Define the linear map $\prime : L(\mathfrak{sl}_2)^+ \to L(\mathfrak{sl}_2)^+$ so that (15) holds. The map has order 3 so it is a bijection. Using (11) we find that the map is a homomorphism of Lie algebras. $\square$

Lemma 6.8 *The following diagram commutes:*

$$\begin{array}{ccc} \boxtimes & \xrightarrow{\sigma} & L(\mathfrak{sl}_2)^+ \\ {\scriptstyle\prime}\big\downarrow & & \big\downarrow{\scriptstyle\prime} \\ \boxtimes & \xrightarrow{\sigma} & L(\mathfrak{sl}_2)^+ \end{array}$$

Proof: Combine the data in Proposition 6.5 with (5), (6). $\square$

Definition 6.9 Recall $L(\mathfrak{sl}_2) = \mathfrak{sl}_2 \otimes \mathbb{K}[T, T^{-1}]$ by Definition 5.1. Also by Definition 6.1 we have $L(\mathfrak{sl}_2)^+ = \mathfrak{sl}_2 \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{K}[T, T^{-1}, (T-1)^{-1}]$. The inclusion map $\mathbb{K}[T, T^{-1}] \to \mathcal{A}$ and the identity map on $\mathfrak{sl}_2$, together induce an injection of Lie algebras $L(\mathfrak{sl}_2) \to L(\mathfrak{sl}_2)^+$. We call this the *natural homomorphism*.

Lemma 6.10 *The following diagram commutes:*

$$\begin{array}{ccc} L(\mathfrak{sl}_2) & \xrightarrow{\text{nat. hom}} & L(\mathfrak{sl}_2)^+ \\ {\scriptstyle\text{st. hom}}\big\downarrow & & \big\downarrow{\scriptstyle\text{id}} \\ \boxtimes & \xrightarrow[\sigma]{} & L(\mathfrak{sl}_2)^+ \end{array}$$

Proof: By Lemma 5.2 and Note 5.4 the following is a generating set for $L(\mathfrak{sl}_2)$:

$$\begin{aligned} e_1 &= e \otimes 1, & f_1 &= f \otimes 1, & h_1 &= h \otimes 1, \\ e_0 &= f \otimes T, & f_0 &= e \otimes T^{-1}, & h_0 &= -h \otimes 1. \end{aligned}$$

We chase these generators around the diagram. We illustrate what happens for the generator e_0. By Lemma 5.3 and Note 5.4 we find

$$e_0 = (X_0 + Z_0)/2. \tag{16}$$

The standard homomorphism from $L(\mathfrak{sl}_2)$ to $\boxtimes$ is the map in Proposition 5.7 with $(h, i, j, k) = (1, 2, 3, 0)$. Applying this map to (16) we get

$$(X_{30} + X_{13})/2. \tag{17}$$

We now apply σ to (17) using Proposition 6.5 and Definition 1.1(i); the result is

$$-(Y + Z) \otimes T/2. \tag{18}$$

Using Lemma 3.2 and Note 3.3 we find (18) is equal to $f \otimes T = e_0$, and this is also the image of e_0 under the natural homomorphism. For the other generators the details are similar and ommitted. $\square$

7 A spanning set for $\boxtimes$

In this section we display a spanning set for $\boxtimes$. Later in the paper it will turn out that this spanning set is a basis for $\boxtimes$.

Definition 7.1 Let Ω denote the subalgebra of $\boxtimes$ generated by X_{12}, X_{03}. We observe that Ω is the image of the Onsager algebra O under the standard homomorphism $O \to \boxtimes$ from Definition 4.9.

We have a comment.

Lemma 7.2 *Ω' is the subalgebra of $\boxtimes$ generated by X_{23}, X_{01}. Ω'' is the subalgebra of $\boxtimes$ generated by X_{31}, X_{02}.*

Definition 7.3 Referring to the standard homomorphism $O \to \boxtimes$ from Definition 4.9, for $m \in \mathbb{Z}$ we let a_m denote the image of A_m. For $l \in \mathbb{N}$ we let g_l denote the image of G_l.

Lemma 7.4 *We have $a_0 = X_{12}$, $a_1 = X_{03}$ and*

$$\begin{aligned} [a_l, a_m] &= 2g_{l-m} \qquad\qquad l > m, \\ [g_l, a_m] &= a_{m+l} - a_{m-l}, \\ [g_l, g_m] &= 0. \end{aligned}$$

Proof: Immediate from Definition 4.1 and Definition 7.3. $\square$

Lemma 7.5 *The following (i)–(iii) hold.*

(i) Ω is spanned by

$$a_m, \; g_l \qquad\qquad m \in \mathbb{Z}, \qquad l \in \mathbb{N}. \tag{19}$$

(ii) Ω' *is spanned by*

$$a'_m,\ g'_l \qquad\qquad m \in \mathbb{Z}, \qquad l \in \mathbb{N}. \tag{20}$$

(iii) Ω'' *is spanned by*

$$a''_m,\ g''_l \qquad\qquad m \in \mathbb{Z}, \qquad l \in \mathbb{N}. \tag{21}$$

Proof: (i) Recall O is spanned by A_m, G_l, $m \in \mathbb{Z}, l \in \mathbb{N}$. Applying the standard homomophism $O \to \boxtimes$ we find Ω is spanned by a_m, g_l, $m \in \mathbb{Z}, l \in \mathbb{N}$.
(ii), (iii) Apply the automorphism $\prime$. $\square$

We are going to prove that the union of (19)–(21) is a spanning set for $\boxtimes$. To do this we show that $\boxtimes = \Omega + \Omega' + \Omega''$. We will use the following lemma.

Lemma 7.6 *For* $m \in \mathbb{N}$,

$$\begin{aligned}
[a'_0, a_m] &= -2a'_0 + 2a_m + 4\sum_{i=1}^{m-1} a_i - 4\sum_{i=1}^{m-1} g_i, & (22)\\
[a'_0, a_{1-m}] &= -2a'_0 - 2a_{1-m} - 4\sum_{i=1}^{m-1} a_{1-i} - 4\sum_{i=1}^{m-1} g_i, & (23)\\
[a'_0, g_m] &= 2\sum_{i=1-m}^{m} a_i & (24)
\end{aligned}$$

and

$$\begin{aligned}
[a'_1, a_m] &= 2a'_1 - 2a_m - 4\sum_{i=1}^{m-1} a_i - 4\sum_{i=1}^{m-1} g_i, & (25)\\
[a'_1, a_{1-m}] &= 2a'_1 + 2a_{1-m} + 4\sum_{i=1}^{m-1} a_{1-i} - 4\sum_{i=1}^{m-1} g_i, & (26)\\
[a'_1, g_m] &= 2\sum_{i=1-m}^{m} a_i. & (27)
\end{aligned}$$

Proof: We first verify (22)–(24) by induction on m. We start with the case $m = 1$. By Lemma 7.4 we have $a_0 = X_{12}$ and $a_1 = X_{03}$. Also $X'_{12} = X_{23}$ by (6) so $a'_0 = X_{23}$. We have $[X_{23}, X_{30}] = 2X_{23} + 2X_{30}$ by Definition 1.1(ii) and $X_{30} = -X_{03}$ by Definition 1.1(i). Combining these comments we find $[a'_0, a_1] = -2a'_0 + 2a_1$ so (22) holds for $m = 1$. We mentioned $a_0 = X_{12}$ and $a'_0 = X_{23}$. We have $[X_{12}, X_{23}] = 2X_{12} + 2X_{23}$ by Definition 1.1(ii) and recall $[X_{12}, X_{23}] = -[X_{23}, X_{12}]$ by the definition of a Lie algebra. Combining these comments we find $[a'_0, a_0] = -2a'_0 - 2a_0$ so (23) holds for $m = 1$. By Lemma 7.4 we find $[a_1, a_0] = 2g_1$. By the Jacobi identity

$$[a'_0, [a_1, a_0]] = [[a'_0, a_1], a_0] + [a_1, [a'_0, a_0]].$$

In this equation we evaluate the left-hand side using $[a_1, a_0] = 2g_1$ and the right-hand side using (22), (23) at $m = 1$. We routinely find $[a'_0, g_1] = 2a_0 + 2a_1$ so (24) holds at $m = 1$. We have now verified (22)–(24) for $m = 1$. Now for an integer $j \geq 2$ we verify (22)–(24) for $m = j$. By induction we may assume (22)–(24) hold for $1 \leq m \leq j - 1$. From Lemma 7.4 we find

$$[g_1, a_{j-1}] = a_j - a_{j-2}. \tag{28}$$

By the Jacobi identity

$$[a'_0, [g_1, a_{j-1}]] = [[a'_0, g_1], a_{j-1}] + [g_1, [a'_0, a_{j-1}]]. \tag{29}$$

In equation (29) we evaluate the left-hand side using (28) and the lines (22), (23) at $1 \leq m \leq j - 1$. Moreover we evaluate the right-hand side using Lemma 7.4 and lines (22)–(24) at $1 \leq m \leq j - 1$. From this we routinely obtain (22) at $m = j$. From Lemma 7.4 we find

$$[g_1, a_{2-j}] = a_{3-j} - a_{1-j}. \tag{30}$$

By the Jacobi identity

$$[a'_0, [g_1, a_{2-j}]] = [[a'_0, g_1], a_{2-j}] + [g_1, [a'_0, a_{2-j}]]. \tag{31}$$

In equation (31) we evaluate the left-hand side using (30) and lines (22), (23) at $1 \leq m \leq j - 1$. Moreover we evaluate the right-hand side using Lemma 7.4 and lines (22)–(24) at $1 \leq m \leq j - 1$. From this we routinely obtain (23) at $m = j$. By Lemma 7.4 we find $[a_j, a_0] = 2g_j$. By the Jacobi identity

$$[a'_0, [a_j, a_0]] = [[a'_0, a_j], a_0] + [a_j, [a'_0, a_0]].$$

In this equation we evaluate the left-hand side using $[a_j, a_0] = 2g_j$ and the right-hand side using (22) at $m = j$. From this we routinely obtain (24) at $m = j$. We have now verified lines (22)–(24). Next we verify lines (25)–(27). To do this we apply the automorphism $\downarrow\Downarrow$ to (22)–(24). Using Lemma 4.5(i),(ii) and Lemma 4.10 we find $a_m^{\downarrow\Downarrow} = -a_m$ for $m \in \mathbb{Z}$ and $g_l^{\downarrow\Downarrow} = g_l$ for $l \in \mathbb{N}$. We now show that $a_0'^{\downarrow\Downarrow} = -a'_1$. We mentioned earlier that $a'_0 = X_{23}$ and $a_1 = X_{03}$. From the former and (4) we find $a_0'^{\downarrow\Downarrow} = X_{10}$. From the latter and (5) we get $a'_1 = X_{01}$. By these remarks and since $X_{10} = -X_{01}$ we find $a_0'^{\downarrow\Downarrow} = -a'_1$. Applying the automorphism $\downarrow\Downarrow$ to (22)–(24) using the above comments we obtain (25)–(27). $\square$

Lemma 7.7 *Each of the following is a subalgebra of* $\boxtimes$*:*

$$\Omega + \Omega', \qquad \Omega' + \Omega'', \qquad \Omega + \Omega''.$$

Proof: We first show that $\Omega + \Omega'$ is a subalgebra of $\boxtimes$. Since each of Ω, Ω' is a subalgebra of $\boxtimes$ it suffices to show $[\Omega, \Omega'] \subseteq \Omega + \Omega'$. Using Lemma 7.6 we find that $\Omega + \Omega'$ is an invariant subspace for $\mathrm{ad}(a'_0)$ and $\mathrm{ad}(a'_1)$. Observe a'_0, a'_1 generate Ω' so $\Omega + \Omega'$ is an invariant subspace for $\mathrm{ad}(\Omega')$. It follows that $[\Omega, \Omega'] \subseteq \Omega + \Omega'$ so $\Omega + \Omega'$ is a subalgebra of $\boxtimes$. Repeatedly applying the automorphism $\prime$ we find that each of $\Omega' + \Omega''$, $\Omega + \Omega''$ is a subalgebra of $\boxtimes$. $\square$

Proposition 7.8 *The following (i), (ii) hold:*

(i) $\boxtimes = \Omega + \Omega' + \Omega''$.

(ii) $\boxtimes$ *is spanned by the union of (19)–(21).*

Proof: (i) By Lemma 7.7 and since each of $\Omega, \Omega', \Omega''$ is a subalgebra of $\boxtimes$ we find $\Omega + \Omega' + \Omega''$ is a subalgebra of $\boxtimes$. This subalgebra contains the generators $\{X_{ij} \mid i, j \in \mathbb{I}, i \neq j\}$ for $\boxtimes$ by Definition 7.1 and Lemma 7.2. Therefore $\boxtimes = \Omega + \Omega' + \Omega''$.
(ii) Combine (i) above with Lemma 7.5. $\square$

Note 7.9 In Section 11 we will show that the sum $\boxtimes = \Omega + \Omega' + \Omega''$ is direct, and that the union of (19)–(21) is a basis for $\boxtimes$.

8 Comments on $L(\mathfrak{sl}_2)^+$

In this section we shift our attention to $L(\mathfrak{sl}_2)^+$. We will define a subalgebra Δ of $L(\mathfrak{sl}_2)^+$ and prove

$$L(\mathfrak{sl}_2)^+ = \Delta + \Delta' + \Delta'' \qquad \text{(direct sum)}.$$

Later in the paper it will turn out that Δ is the image of Ω under the map σ from Proposition 6.5.

Before proceeding we sharpen our notation. Referring to Definition 6.1 and Lemma 6.2, for $S \in \{T, T', T''\}$ we identify $\mathbb{K}[S]$ with the subalgebra of $\mathcal{A}$ generated by S.

Definition 8.1 We let Δ denote the following subspace of $L(\mathfrak{sl}_2)^+$:

$$\Delta = X \otimes \mathbb{K}[T] + Y \otimes T\mathbb{K}[T] + Z \otimes (T-1)\mathbb{K}[T]. \tag{32}$$

Lemma 8.2 *We have*

$$\begin{aligned}
\Delta' &= X \otimes (T'-1)\mathbb{K}[T'] + Y \otimes \mathbb{K}[T'] + Z \otimes T'\mathbb{K}[T'], \\
\Delta'' &= X \otimes T''\mathbb{K}[T''] + Y \otimes (T''-1)\mathbb{K}[T''] + Z \otimes \mathbb{K}[T''].
\end{aligned}$$

Proof: Routine using Lemma 6.7 and Lemma 3.4(i). $\square$

Lemma 8.3 *Each of* $\Delta, \Delta', \Delta''$ *is a subalgebra of* $L(\mathfrak{sl}_2)^+$.

Proof: Using (7) and (11) we find Δ is closed under the Lie bracket. Therefore Δ is a subalgebra of $L(\mathfrak{sl}_2)^+$. By this and since the map $'$ is an automorphism we find each of Δ', Δ'' is a subalgebra of $L(\mathfrak{sl}_2)^+$. $\square$

Proposition 8.4 *We have*

$$L(\mathfrak{sl}_2)^+ = \Delta + \Delta' + \Delta'' \qquad \text{(direct sum)}.$$

Proof: The elements X, Y, Z form a basis for $\mathfrak{sl}_2$ so

$$L(\mathfrak{sl}_2)^+ = X \otimes \mathcal{A} + Y \otimes \mathcal{A} + Z \otimes \mathcal{A} \qquad \text{(direct sum)}.$$

From Lemma 6.3 we find

$$\mathcal{A} = \mathbb{K}[T] + (T' - 1)\mathbb{K}[T'] + T''\mathbb{K}[T''] \qquad \text{(direct sum)}$$

and this implies

$$X \otimes \mathcal{A} = X \otimes \mathbb{K}[T] + X \otimes (T' - 1)\mathbb{K}[T'] + X \otimes T''\mathbb{K}[T''] \qquad \text{(direct sum)}. \tag{33}$$

We apply the automorphism $\prime$ to (33) and in the resulting equation cyclically permute the terms on the right-hand side. This gives

$$Y \otimes \mathcal{A} \;=\; Y \otimes T\mathbb{K}[T] + Y \otimes \mathbb{K}[T'] + Y \otimes (T'' - 1)\mathbb{K}[T''] \qquad \text{(direct sum)}. \tag{34}$$

We apply the automorphism $\prime$ to (34) and in the resulting equation cyclically permute the terms on the right-hand side. This gives

$$Z \otimes \mathcal{A} \;=\; Z \otimes (T - 1)\mathbb{K}[T] + Z \otimes T'\mathbb{K}[T'] + Z \otimes \mathbb{K}[T''] \qquad \text{(direct sum)}. \tag{35}$$

Using Definition 8.1 we find Δ is the sum of the first terms on the right in lines (33), (34), (35). Similarly Δ' (resp. Δ'') is the sum of the second terms (resp. third terms) on the right in lines (33), (34), (35). The result follows. $\square$

9 Some polynomials

In this section we recall the Chebyshev polynomials. We will use the following notation. Let λ denote an indeterminate. Let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{K}$.

Definition 9.1 [4, p. 101] For an integer $n \geq 0$ we let U_n denote the polynomial in $\mathbb{K}[\lambda]$ that satisfies

$$U_n\Big(\frac{\lambda + \lambda^{-1}}{2}\Big) = \frac{\lambda^{n+1} - \lambda^{-n-1}}{\lambda - \lambda^{-1}}.$$

We call U_n the *nth Chebyshev polynomial of the second kind.*

Example 9.2 We have

$$U_0 = 1, \qquad U_1 = 2\lambda, \qquad U_2 = 4\lambda^2 - 1, \qquad U_3 = 8\lambda^3 - 4\lambda,$$
$$U_4 = 16\lambda^4 - 12\lambda^2 + 1, \qquad U_5 = 32\lambda^5 - 32\lambda^3 + 6\lambda.$$

Lemma 9.3 *[29, Section 1.8.2] The Chebyshev polynomials satisfy the following 3-term recurrence:*

$$2\lambda U_n = U_{n+1} + U_{n-1} \qquad n = 0, 1, \ldots$$
$$U_0 = 1, \qquad U_{-1} = 0.$$

Lemma 9.4 *[29, Section 1.8.2] The Chebyshev polynomials have the following presentation in terms of hypergeometric series:*

$$U_n(\lambda) = (n+1)\, {}_2F_1\left(\begin{matrix} -n,\, n+2 \\ 3/2 \end{matrix} \,\middle|\, \frac{1-\lambda}{2} \right) \qquad\qquad n = 0, 1, 2, \ldots$$

We have a comment.

Lemma 9.5 *The following is a basis for the $\mathbb{K}$-vector space $\mathbb{K}[\lambda]$:*

$$U_n(1-2\lambda) \qquad\qquad n = 0, 1, 2, \ldots$$

Proof: For an integer $n \geq 0$ the polynomial U_n has degree exactly n by Lemma 9.3. From this we find $U_n(1-2\lambda)$ has degree exactly n as a polynomial in λ. The result follows. $\square$

10 A basis for $L(\mathfrak{sl}_2)^+$

In Section 8 we obtained a direct sum decompostion $L(\mathfrak{sl}_2)^+ = \Delta + \Delta' + \Delta''$. In this section we find a basis for each of $\Delta, \Delta', \Delta''$. The union of these bases is a basis for $L(\mathfrak{sl}_2)^+$ that we will find useful later in the paper.

Lemma 10.1 *Referring to Definition 8.1 the following (i)–(iv) hold.*

(i) $X \otimes \mathbb{K}[T]$ *has a basis*

$$X \otimes U_{m-1}(1-2T) \qquad\qquad m \in \mathbb{N}. \tag{36}$$

(ii) $Y \otimes T\mathbb{K}[T]$ *has a basis*

$$Y \otimes TU_{m-1}(1-2T) \qquad\qquad m \in \mathbb{N}. \tag{37}$$

(iii) $Z \otimes (T-1)\mathbb{K}[T]$ *has a basis*

$$Z \otimes (T-1)U_{m-1}(1-2T) \qquad\qquad m \in \mathbb{N}. \tag{38}$$

(iv) The union of (36)–(38) is a basis for Δ.

Proof: The assertions (i)–(iii) follow from Lemma 9.5. Assertion (iv) follows from (i)–(iii) and since the sum (32) is direct. $\square$

Lemma 10.2 *The following (i)–(iv) hold.*

(i) $X \otimes (T'-1)\mathbb{K}[T']$ *has a basis*

$$X \otimes (T'-1)U_{m-1}(1-2T') \qquad\qquad m \in \mathbb{N}. \tag{39}$$

(ii) $Y \otimes \mathbb{K}[T']$ *has a basis*

$$Y \otimes U_{m-1}(1-2T') \qquad\qquad m \in \mathbb{N}. \tag{40}$$

(iii) $Z \otimes T'\mathbb{K}[T']$ *has a basis*

$$Z \otimes T'U_{m-1}(1-2T') \qquad\qquad m \in \mathbb{N}. \tag{41}$$

(iv) The union of (39)–(41) is a basis for Δ'*.*

Proof: Apply the automorphism $\prime$ to the vectors in Lemma 10.1. $\square$

Lemma 10.3 *The following (i)–(iv) hold.*

(i) $X \otimes T''\mathbb{K}[T'']$ *has a basis*

$$X \otimes T''U_{m-1}(1-2T'') \qquad\qquad m \in \mathbb{N}. \tag{42}$$

(ii) $Y \otimes (T''-1)\mathbb{K}[T'']$ *has a basis*

$$Y \otimes (T''-1)U_{m-1}(1-2T'') \qquad\qquad m \in \mathbb{N}. \tag{43}$$

(iii) $Z \otimes \mathbb{K}[T'']$ *has a basis*

$$Z \otimes U_{m-1}(1-2T'') \qquad\qquad m \in \mathbb{N}. \tag{44}$$

(iv) The union of (42)–(44) is a basis for Δ''*.*

Proof: Apply the automorphism $\prime$ to the vectors in Lemma 10.2. $\square$

Theorem 10.4 *The union of (36)–(44) is a basis for the* $\mathbb{K}$*-vector space* $L(\mathfrak{sl}_2)^+$*.*

Proof: Combine Proposition 8.4 with Lemma 10.1(iv), Lemma 10.2(iv), Lemma 10.3(iv). $\square$

11 The main results

In this section we show that the sum $\boxtimes = \Omega + \Omega' + \Omega''$ is direct, and that the union of (19)–(21) is a basis for $\boxtimes$. We also show that the Lie algebra homomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$ from Proposition 6.5 is an isomorphism. Our proofs are based on the following proposition, in which we apply σ to (19)–(21) and express the image in terms of the basis from Theorem 10.4.

Proposition 11.1 *Let the homomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$ be as in Proposition 6.5. Then for each element u in the table below, the expression to the right of u is the image of u under σ.*

element u	image of u under σ
a_m	$-X \otimes U_{m-2}(1-2T) + Y \otimes TU_{m-1}(1-2T) + Z \otimes (T-1)U_{m-1}(1-2T)$
a_{1-m}	$X \otimes U_{m-1}(1-2T) - Y \otimes TU_{m-2}(1-2T) - Z \otimes (T-1)U_{m-2}(1-2T)$
g_m	$-X \otimes U_{m-1}(1-2T) - Y \otimes TU_{m-1}(1-2T) + Z \otimes (T-1)U_{m-1}(1-2T)$
a'_m	$X \otimes (T'-1)U_{m-1}(1-2T') - Y \otimes U_{m-2}(1-2T') + Z \otimes T'U_{m-1}(1-2T')$
a'_{1-m}	$-X \otimes (T'-1)U_{m-2}(1-2T') + Y \otimes U_{m-1}(1-2T') - Z \otimes T'U_{m-2}(1-2T')$
g'_m	$X \otimes (T'-1)U_{m-1}(1-2T') - Y \otimes U_{m-1}(1-2T') - Z \otimes T'U_{m-1}(1-2T')$
a''_m	$X \otimes T''U_{m-1}(1-2T'') + Y \otimes (T''-1)U_{m-1}(1-2T'') - Z \otimes U_{m-2}(1-2T'')$
a''_{1-m}	$-X \otimes T''U_{m-2}(1-2T'') - Y \otimes (T''-1)U_{m-2}(1-2T'') + Z \otimes U_{m-1}(1-2T'')$
g''_m	$-X \otimes T''U_{m-1}(1-2T'') + Y \otimes (T''-1)U_{m-1}(1-2T'') - Z \otimes U_{m-1}(1-2T'')$

In the above table we assume $m \in \mathbb{N}$.

Proof: Referring to the above table, for $m \in \mathbb{N}$ let $\hat{a}_m$, $\hat{a}_{1-m}$, and $\hat{g}_m$ denote the expressions to the right of a_m, a_{1-m}, and g_m respectively. We show $a_m^\sigma = \hat{a}_m$, $a_{1-m}^\sigma = \hat{a}_{1-m}$, and $g_m^\sigma = \hat{g}_m$. Using the data in the above table we find

$$\hat{a}_0 = X \otimes 1, \qquad \hat{a}_1 = Y \otimes T + Z \otimes (T-1).$$

Using the data in the above table and (7), Lemma 9.3 we find

$$\begin{aligned} [\hat{a}_m, \hat{a}_0] &= 2\hat{g}_m, \\ [\hat{g}_1, \hat{a}_m] &= \hat{a}_{m+1} - \hat{a}_{m-1}, \\ [\hat{g}_1, \hat{a}_{1-m}] &= \hat{a}_{2-m} - \hat{a}_{-m} \end{aligned}$$

for $m \in \mathbb{N}$. Recall by Lemma 7.4 that $a_0 = X_{12}$, $a_1 = X_{03}$. By this and Proposition 6.5 we find

$$a_0^\sigma = X \otimes 1, \qquad a_1^\sigma = Y \otimes T + Z \otimes (T-1).$$

By Lemma 7.4 and since σ is a homomorphism of Lie algeras,

$$\begin{aligned} [a_m^\sigma, a_0^\sigma] &= 2g_m^\sigma, \\ [g_1^\sigma, a_m^\sigma] &= a_{m+1}^\sigma - a_{m-1}^\sigma, \\ [g_1^\sigma, a_{1-m}^\sigma] &= a_{2-m}^\sigma - a_{-m}^\sigma \end{aligned}$$

for $m \in \mathbb{N}$. By these comments the $\hat{a}_m, \hat{a}_{1-m}, \hat{g}_m$ and the $a_m^\sigma, a_{1-m}^\sigma, g_m^\sigma$ satisfy the same recursion and the same initial conditions. It follows $a_m^\sigma = \hat{a}_m$, $a_{1-m}^\sigma = \hat{a}_{1-m}$, and $g_m^\sigma = \hat{g}_m$ for $m \in \mathbb{N}$. We have now verified the upper third of the table. To verify the remaining two thirds of the table, apply the automorphism $\prime$ and use Lemma 6.8. $\square$

Lemma 11.2 *The following (i)–(iv) hold.*

(i) Δ has a basis

$$a_m^\sigma,\ g_l^\sigma \qquad\qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}. \tag{45}$$

(ii) Δ' has a basis

$$a_m'^\sigma,\ g_l'^\sigma \qquad\qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}. \tag{46}$$

(iii) Δ'' has a basis

$$a_m''^\sigma,\ g_l''^\sigma \qquad\qquad m \in \mathbb{Z}, \quad l \in \mathbb{N}. \tag{47}$$

(iv) The union of (45)–(47) is a basis for $L(\mathfrak{sl}_2)^+$.

Proof: (i) The elements (45) are contained in Δ by Definition 8.1 and the data in the table of Proposition 11.1. In Lemma 10.1(iv) we gave a basis for Δ. Consider the following ordering of the vectors in this basis:

$$\begin{aligned} X \otimes 1, \quad Y \otimes T, \quad Z \otimes (T-1), \quad X \otimes U_1(1-2T),& \\ Y \otimes TU_1(1-2T), \quad Z \otimes (T-1)U_1(1-2T), \quad \ldots& \end{aligned} \tag{48}$$

Now consider the sequence

$$a_0^\sigma, \quad a_1^\sigma - g_1^\sigma, \quad g_1^\sigma, \quad a_{-1}^\sigma, \quad a_2^\sigma - g_2^\sigma, \quad g_2^\sigma, \quad \ldots \tag{49}$$

Using the table in Proposition 11.1 we express each vector in (49) as a linear combination of (48). We observe that the corresponding matrix of coefficients is upper triangular with all diagonal entries nonzero. By this and since (48) is a basis for Δ we find (49) is a basis for Δ. From this we routinely find that (45) is a basis for Δ.
(ii), (iii) Apply the automorphism $\prime$ and use Lemma 6.8.
(iv) Use Proposition 8.4 and (i)–(iii) above. $\square$

Corollary 11.3 *Under the map $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$ from Proposition 6.5, the image of $\Omega, \Omega', \Omega''$ is $\Delta, \Delta', \Delta''$ respectively.*

Proof: We first show that Δ is the image Ω^σ. The vectors a_m, g_l, $m \in \mathbb{Z}, l \in \mathbb{N}$ span Ω by Lemma 7.5(i). Therefore the vectors a_m^σ, g_l^σ, $m \in \mathbb{Z}, l \in \mathbb{N}$ span Ω^σ. But these vectors span Δ by Lemma 11.2(i) so $\Omega^\sigma = \Delta$. We have now shown that Δ is the image of Ω under σ. Our remaining assertions follow from this and Lemma 6.8. $\square$

Theorem 11.4 *The following (i)–(iv) hold.*

(i) The elements (19) form a basis for Ω.

(ii) The elements (20) form a basis for Ω'.

(iii) The elements (21) form a basis for Ω''.

(iv) The union of (19)–(21) is a basis for $\boxtimes$.

Proof: (i) The elements (19) span Ω by Lemma 7.5(i). The elements (19) are linearly independent since their images under σ are linearly independent by Lemma 11.2(i).
(ii), (iii) Similar to the proof of (i) above.
(iv) The vectors (19)–(21) span $\boxtimes$ by Proposition 7.8. The vectors (19)–(21) are linearly independent since their images under σ are linearly independent by Lemma 11.2(iv). $\square$

Theorem 11.5 *The Lie algebra homomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$ from Proposition 6.5 is an isomorphism.*

Proof: The map σ sends the basis for $\boxtimes$ given in Theorem 11.4(iv) to the basis for $L(\mathfrak{sl}_2)^+$ given in Lemma 11.2(iv). The result follows. $\square$

Theorem 11.6 *The sum $\boxtimes = \Omega + \Omega' + \Omega''$ is direct.*

Proof: Immediate from Theorem 11.4. $\square$

12 Conclusion

In this section we prove the outstanding assertions from earlier in the paper.

Corollary 12.1 *The Lie algebra homomorphism $\mathfrak{sl}_2 \to \boxtimes$ given in Proposition 3.6 is an injection.*

Proof: Referring to Proposition 3.6 and in view of the S_4-action on $\boxtimes$, without loss we may assume $(h, i, j) = (1, 2, 3)$ so that the homomorphism is standard. The elements X, Y, Z form a basis for $\mathfrak{sl}_2$, and their images under the standard homomorphism are X_{12}, X_{23}, X_{31} respectively. It suffices to show that X_{12}, X_{23}, X_{31} are linearly independent. They are linearly independent since $X_{12} = a_0$, $X_{23} = a'_0$, $X_{31} = a''_0$ and since a_0, a'_0, a''_0 are linearly independent by Theorem 11.4(iv). $\square$

Corollary 12.2 *The Lie algebra homomorphism $O \to \boxtimes$ given in Proposition 4.7 is an injection.*

Proof: With reference to Proposition 4.7 and in view of the S_4-action on $\boxtimes$, without loss we may assume $(h, i, j, k) = (1, 2, 0, 3)$ so that the homomorphism is standard. The elements A_m, G_l, $m \in \mathbb{Z}, l \in \mathbb{N}$ form a basis for O and their images under the standard homomorphism are a_m, g_l, $m \in \mathbb{Z}, l \in \mathbb{N}$. Therefore it suffices to show that a_m, g_l, $m \in \mathbb{Z}, l \in \mathbb{N}$ are linearly independent. But this is the case by Theorem 11.4(i). $\square$

Corollary 12.3 *The Lie algebra homomorphism $L(\mathfrak{sl}_2) \to \boxtimes$ given in Proposition 5.7 is an injection.*

Proof: Referring to Proposition 5.7 and in view of the S_4-action on $\boxtimes$, without loss we may assume $(h, i, j, k) = (1, 2, 3, 0)$ so that the homomorphism is standard. By Definition 6.9 the natural homomorphism $L(\mathfrak{sl}_2) \to L(\mathfrak{sl}_2)^+$ is an injection. By Lemma 6.10 the natural homomorphism is the composition of the standard homomorphism $L(\mathfrak{sl}_2) \to \boxtimes$ and the isomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$. Therefore the standard homomorphism $L(\mathfrak{sl}_2) \to \boxtimes$ is an injection. The result follows. $\square$

Corollary 12.4 *For mutually distinct $h, i, j \in \mathbb{I}$ the elements X_{hi}, X_{ij}, X_{jh} form a basis for a subalgebra of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$.*

Proof: Immediate from Proposition 3.6 and Corollary 12.1. $\square$

Corollary 12.5 *For mutually distinct $h, i, j, k \in \mathbb{I}$ the subalgebra of $\boxtimes$ generated by X_{hi}, X_{jk} is isomorphic to the Onsager algebra.*

Proof: Immediate from Proposition 4.7 and Corollary 12.2. $\square$

Corollary 12.6 *For distinct $r, s \in \mathbb{I}$ the subalgebra of $\boxtimes$ generated by*

$$\{X_{ij} \,|\, i, j \in \mathbb{I}, i \neq j, (i, j) \neq (r, s), (i, j) \neq (s, r)\}$$

is isomorphic to the loop algebra $L(\mathfrak{sl}_2)$.

Proof: This subalgebra is the image of $L(\mathfrak{sl}_2)$ under the homomorphism in Proposition 5.7, where in that proposition we take $i = r$ and $k = s$. The result follows in view of Corollary 12.3. $\square$

We finish this section with a comment.

Corollary 12.7 *The group homomorphism $S_4 \to Aut(\boxtimes)$ from Section 2 is an injection.*

Proof: The elements

$$X_{12}, X_{23}, X_{31}, X_{03}, X_{01}, X_{02}$$

are linearly independent since they are the images under the isomorphism σ of a_0, a_0', a_0'', a_1, a_1', a_1''. Combining this with Definition 1.1(i) we find $\{X_{ij} \,|\, i, j \in \mathbb{I}, i \neq j\}$ are mutually distinct. For τ in the kernel of the group homomorphism $S_4 \to \mathrm{Aut}(\boxtimes)$ and for distinct $i, j \in \mathbb{I}$ we find $X_{ij} = X_{i^\tau j^\tau}$ by (2), so $i^\tau = i$ and $j^\tau = j$ by our preliminary remark. Apparently τ stabilizes each element of $\mathbb{I}$ so τ is the identity element. The result follows. $\square$

13 Suggestions for further research

In this section we give some suggestions for further research.

Problem 13.1 By Theorem 11.4 the union of (19)–(21) is a basis for the $\mathbb{K}$-vector space $\boxtimes$. Compute the action of the Lie bracket on this basis. See Lemma 7.4 and Lemma 7.6 for partial results.

Problem 13.2 Compute the group $\mathrm{Aut}(\boxtimes)$. We recall by Corollary 12.7 that the homomorphism of groups $S_4 \to \mathrm{Aut}(\boxtimes)$ given in Section 2 is an injection.

Problem 13.3 Find all the ideals in the Lie algebra $\boxtimes$.

The following problem was inspired by [18].

Problem 13.4 For $i, j \in \{0, 1\}$ we define

$$\boxtimes_{ij} = \{v \in \boxtimes \mid v^d = (-1)^i v, \quad v^* = (-1)^j v\}$$

where the automorphisms $d, *$ are from (3), (4) respectively. Since $d, *$ are commuting involutions we find

$$\boxtimes = \boxtimes_{00} + \boxtimes_{01} + \boxtimes_{10} + \boxtimes_{11} \qquad \text{(direct sum)}.$$

By the construction

$$[\boxtimes_{ij}, \boxtimes_{rs}] \subseteq \boxtimes_{i+r, j+s} \tag{50}$$

where the subscripts are computed modulo 2. Show that $\boxtimes_{00} = 0$, and conclude using (50) that each of $\boxtimes_{01}, \boxtimes_{10}, \boxtimes_{11}$ is an abelian subalgebra of $\boxtimes$. Find a basis for each of these subalgebras. Investigate the relationship between the decomposition $\boxtimes = \boxtimes_{01} + \boxtimes_{10} + \boxtimes_{11}$ and the decomposition $\boxtimes = \Omega + \Omega' + \Omega''$ from Theorem 11.6.

Given the results of this paper it is natural to consider the following generalization of the algebra $\boxtimes$.

Problem 13.5 By a *graph* we mean a pair $\Gamma = (X, E)$ where X is a nonempty finite set and $E \subseteq X^2$ is a binary relation such that $ii \notin E$ for all $i \in X$ and $ij \in E \Leftrightarrow ji \in E$ for all $i, j \in X$. Given a graph $\Gamma = (X, E)$ let $\mathcal{L} = \mathcal{L}(\Gamma)$ denote the Lie algebra over $\mathbb{K}$ that has generators

$$\{X_{ij} \mid i, j \in X, \quad ij \in E\}$$

and the following relations:

(i) For $ij \in E$,

$$X_{ij} + X_{ji} = 0.$$

(ii) For $hi \in E$ and $ij \in E$,

$$[X_{hi}, X_{ij}] = 2X_{hi} + 2X_{ij}.$$

(iii) For $hi \in E$ and $jk \in E$,

$$[X_{hi}, [X_{hi}, [X_{hi}, X_{jk}]]] = 4[X_{hi}, X_{jk}].$$

Given a subset $E_1 \subseteq E$ the inclusion map $E_1 \to E$ induces a homomorphism of Lie algebras $\mathcal{L}(\Gamma_1) \to \mathcal{L}(\Gamma)$, where $\Gamma_1 = (X, E_1)$. Show that this homomorphism is an injection.

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